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ON THE STABILITY OF INHOMOGENEOUS STEADY STATES IN A NONEQUILIBRIUM MAGNETIZED PLASMA

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Stability of inhomogeneous steady distributions of the electron temperature (electron concentration) and the electrodynamic parameters (current density and electric field) in a channel, relative to the one-dimensional perturbations, is investigated. A criterion of stability is obtained for a layered wave.

The authors of [1] have shown that distributions of electron concentrations may exist in a channel with a nonequilibrium magnetized plasma, representing homogeneous regions separated by stationary surfaces of discontinuity (layered waves). Stationary solitons (solitary waves) may appear for a specified composition of the plasma in the channel, at various values of the potential difference at the electrodes. Problems exist in which the solution with travelling layered waves or solitons may be utilized. Analogous distributions of the parameters of the medium arise in the problems concerned with semiconductors [2,3] and gas discharge in a nonequilibrium plasma [4—6]. The present paper deals with the stability of such inhomogeneous steady states under one-dimensional perturbations and shows, that the conclusions of [7] need to be refined.

Let is investigate the spectrum of the linear problem of stability of an inhomogeneous state corresponding to a standing layered wave [1] of thickness $2l_1$ symmetrically distributed with respect to the channel center ($S_0(x)$ is the dimensionless electron concentration, and the plane of the wave is parallel to the electrode surfaces)

$$S_0(x) = \begin{cases} S_1 = \text{const}, & -1 \leqslant x \leqslant -l_1,\ l_1 \leqslant x \leqslant 1 \\ S_2 = \text{const}, & l_2 \leqslant x \leqslant l_1,\ S_1 \neq S_3 \end{cases}$$

(the discontinuous solution will be regarded as the limit of the continuous solution in the channel).

The dimensionless electron concentration satisfies the equation

$$\frac{d}{dx} \Lambda(S_0) \frac{dS_0}{dx} + U(S_0) \frac{dS_0}{dx} + F(S_0) = 0$$

the boundary condition

$$S_0(-1) = S_0(1)$$

and the condition of symmetry with respect to the channel center. The coordinate origin coincides with the channel center. The electrode surfaces extended to infinity are parallel to the $y0z$ -plane. The y -axis is directed along the channel, and the magnetic field along the z -axis. The distance $2l$ between the electrodes serves as the characteristic dimension. The remaining notation follows that of [1]. Only dimensionless variables will be used. The analysis of the spectrum yields conclusions concerning the stability of the stationary soliton and of the states with moving layered waves or solitons. The results obtained can also be used in the study of the influence of the external electrical network of the installation on the stability of the inhomogeneous states.

The system of equations for one-dimensional perturbations $S^+(x,t)$ can be written in the form

$$\partial S^+/\partial t = LS^+ \tag{1}$$

$$E_y \equiv E_0 = \text{const}, \quad j_x \equiv j_0 = \text{const}, \quad F = F(E_0,\ j_0,\ S_0(x))$$

$$L \equiv \Lambda(S_0(x))\frac{\partial^2}{\partial x^2} + U(S_0(x))\frac{\partial}{\partial x} + F_s'(S_0(x)) +$$

$$U_s'(S_0(x))\frac{dS_0}{dx} + \Lambda_s'(S_0(x))\frac{d^2S_0}{dx^2}$$

The following relations serve as the boundary conditions for the perturbations S^+:

$$S^+(-1,\ t) = S^+(1,\ t) = 0 \tag{2}$$

Performing the transformation

$$\chi^+(x,t) = S^+(x,t)\exp\left\{-\frac{1}{2}\int_{-1}^{x} U(S_0(\zeta))\Lambda^{-1}d\zeta\right\}$$

we reduce (1) and the boundary conditions (2) to the form (H is a selfconjugate operator and $\delta(x)$ denotes the delta function)

$$\partial\chi^+/\partial t = H\chi^+, \quad \chi^+(-1,\ t) = \chi^+(1,\ t) = 0 \tag{3}$$

$$H \equiv \Lambda(x)\frac{\partial^2}{\partial x^2} - f(x) + F_2'[\delta(x+l_1) + \delta(x-l_1)]$$

$$f(x) = \begin{cases} F_1' = \text{const} > 0, & -1 \leqslant x < -l_1,\ l_1 < x \leqslant 1 \\ F_3' = \text{const} > 0, & -l_1 < x < l_1 \end{cases} \tag{4}$$

$$\Lambda(x) = \begin{cases} \Lambda_1 = \text{const} > 0, & -1 \leqslant x < -l_1,\ l_1 < x \leqslant 1 \\ \Lambda_2 = \text{const} > 0, & -l_1 < x < l_1 \end{cases}$$

$$F_2' = \text{const} > 0$$

Setting $\chi^+(x,\ t) = e^{-pt}\chi(x)$, we reduce the problem of stability of the initial state $S_0(x)$ to that of finding the eigenvalues of the operator H

$$(H+p)\chi = 0, \quad \chi(-1) = \chi(1) = 0 \tag{5}$$

Let us consider separately the spectra of the perturbations, symmetric $\chi^{(s)}(x) = \chi^{(s)}(-x)$ and antisymmetric $\chi_{(a)}(x) = -\chi_{(a)}(-x)$ with respect to the channel center

$$\chi^{(s)}(x) = \begin{cases} \chi^{(1)}(x); & -1 \leqslant x \leqslant -l_1 \\ \chi^{(2)}(x), & -l_1 \leqslant x \leqslant l_1, \\ \chi^{(3)}(x), & l_1 \leqslant x \leqslant 1 \end{cases} \qquad \chi_{(a)}(x) = \begin{cases} \chi_{(1)}(x), & -1 \leqslant x \leqslant -l_1 \\ \chi_{(2)}(x), & -l_1 \leqslant x \leqslant l_1 \\ \chi_{(3)}(x), & l_1 \leqslant x \leqslant 1 \end{cases}$$

Utilizing the results of the theorems given in [8, 9], we find $p_0 < p_1 < \cdots < p_k < \ldots$. The eigenfunction χ_k corresponding to the eigenvalue p_k has k zeros in the interval $(-1, +1)$. Using (4), we obtain the following respective equations for determining the eigenvalues of the symmetric and antisymmetric perturbations:

$$\varkappa_1 \operatorname{ch}\varkappa_3 l_1 \operatorname{ch}\varkappa_1(1-l_1) + \varkappa_3 \operatorname{sh}\varkappa_3 l_1$$
$$\operatorname{sh}\varkappa_1(1-l_1) = F_2' \operatorname{sh}\varkappa_1(1-l_1)\operatorname{ch}\varkappa_3 l_1$$
$$F_2' \operatorname{sh}\varkappa_1(1-l_1)\operatorname{sh}\varkappa_3 l_1 = \varkappa_1 \operatorname{ch}\varkappa_1(1-l_1)\operatorname{sh}\varkappa_3 l_1 +$$
$$\varkappa_3 \operatorname{sh}\varkappa_1(1-l_1)\operatorname{ch}\varkappa_3 l_1$$

$$\varkappa_1 = \frac{F_1' - p}{\Lambda_1}, \quad \varkappa_3 = \frac{F_3' - p}{\Lambda_3}, \quad F_k' \equiv F_s'(S_k)$$
$$\Lambda_k = \Lambda(S_k), \quad k = 1, 3$$

A simpler case is obtained by setting

$$\varkappa_1 = \varkappa_3 = \varkappa, \quad \varkappa = \min(\varkappa_1, \varkappa_3)$$

and the conclusions derived from it remain valid in the general case. The equations for the eigenvalues of the symmetric and antisymmetric perturbations are, respectively,

$$\frac{\varkappa}{F_2'} = \frac{\operatorname{sh}\varkappa(1 - l_1)\operatorname{ch}\varkappa l_1}{\operatorname{ch}\varkappa} \tag{6}$$

$$\frac{\varkappa}{F_2'} = -\frac{\operatorname{sh}\varkappa(1 - l_1)\operatorname{sh}\varkappa l_1}{\operatorname{sh}\varkappa} \tag{7}$$

Equation (6) which is convenient to use in graphical investigations, implies the existence of not more than one negative eigenvalue p_0 corresponding to the perturbation χ_0 nonvanishing anywhere in the interval $(-1, +1)$. The remaining eigenvalues for the symmetric perturbations are all positive. Using (7) we can show, that the eigenvalues of the antisymmetric perturbations are nonnegative, and $p_1 \geqslant 0$ corresponds to the perturbation χ_1 which has a single zero in the interval$(-1, +1)$. The condition of the negativeness of p_0 can be written in the form

$$F_2' > \sqrt{\frac{4F'}{\Lambda}}, \quad \frac{F'}{\Lambda} = \min\left(\frac{F_1'}{\Lambda_1}, \frac{F_3'}{\Lambda_3}\right) \tag{8}$$

The appearance of a negative eigenvalue is connected with the following physical fact. A layered wave appears when the Hall parameters exceed a critical one, and the equation balancing the Joule heating and the energy transfer between the electrons and heavy particles has ambiguous solution (function F has three roots). The layered wave corresponds to a discontinuous passage from the stable state $(S_1, F_s'(S_1) < 0)$ to the stable state $(S_3, F_s'(S_3) < 0)$. Although the wave structure degenerates into an infinitely thin surface when $x = \pm l_1$, it corresponds to a continuous passage from the point S_1 to the point S_3 and it therefore contains the values of temperature at which $F_s' > 0$ and the perturbations are unstable. The condition (8) must hold if the whole system consisting of the stable phases

$$S_1\,(-1 \leqslant x < -l_1,\ l_1 < x \leqslant 1),\ S_3(-l_1 < x < l_1)$$

and the unstable phase S_3 (at $x = -l_1$ and $x = l_1$) is to be unstable.

The conclusions derived from (6) and (7) concerning the position of the spectral points coincide with the following qualitative investigation given in [7]. It can be shown that the function dS_0/dx satisfies the equation $L\,(dS_0/dx) = 0$ and corresponds to the eigenvalue $p = 0$. Strictly speaking, dS_0/dx does not satisfy the boundary conditions (2), but

$$dS_0/dx|_{x=\pm 1} \to 0$$

with the increasing dimensions of the channel. We can therefore assume that one of the eigenfunctions χ_p is proportional to $dS_0/dx : \chi_p = \gamma\,(x) dS_0/dx$. Knowing the behavior of the functions $S_0\,(x)$ and dS_0/dx in the interval $(-1, +1)$ (i.e. the number of the extrema of S_0), we can arrive at certain conclusions concerning the fact whether or not, $p = 0$ is the smallest eigenvalue. If $p = 0$ is not the smallest eigenvalue, then there exists at least one negative eigenvalue $p_0 < 0$. Since S_0 corresponding to the layered wave represents a discontinuous solution (dS_0/dx vanishes for the whole interval of values of the variable x), then we must estimate the smallest eigenvalue using the strict results following from (6) and (7), Indeed, if the inequality (8) does not hold, then the perturbation spectrum will not contain any negative eigenvalues irrespective of the fact that the initial state with the layered wave S_0 can be referred (as was done in some of the problems in [3]) to a solution with a single extremum in the channel.

If the stationary state S_0 is a soliton symmetrical with respect to the channel center (coordinate origin), then the function S_0 will have a single extremum ($dS_0/dx = 0$ when $x = 0$). Consequently $p_1 = 0$ is the eigenvalue of the antisymmetric perturbation $\chi_1 = \gamma(x)\, dS_0/dx$, with a single zero in the interval $(-1, +1)$. In this case there exists a symmetric eigenfunction χ_0 which has no zeros on the above interval.

Function χ_0 has a corresponding negative eigenvalue $p_0 < 0$. If the stationary state represents several solitons, then several negative eigenvalues will exist. Obviously, the presence of a negative eigenvalue in the intial state corresponding to a soliton is connected with the appearance of an interval of values of x in which $F_s' > 0$. More detailed discussion based on theorems of [8, 9] leads to a conclusion that the modulus of the negative eigenvalue depends on the width of the soliton (in the case when a single soliton is present).

In the case of a one-dimensional layered wave and a soliton in motion, the function F has the same form as in the case of the stationary waves, therefore the results concerning the position of the points of the spectrum of a one-dimensional perturbation remain valid for the solutions which are periodic with respect to the variable $\xi = x + Wt$ (with the period equal to the dimension of the channel). Since in the case of a single layered wave or of a single soliton not more than a single negative eigenvalue can exist, the study of the influence of the external electrical network of the device on the stability, becomes sufficiently simple.

Taking into account the non-dimensional perturbations $\sim \chi(x, t) \exp i(K_y y + K_z z)$ of the stationary layered wave we find, that the neutral curve separating the regions of stability and instability will be a function of the parameters Ω, F', Λ, $K = \sqrt{K_y^2 + K_z^2}$, K_z / K_y. A critical value of the Hall parameter Ω^+ exists, on exceeding which the layered wave, with a surface parallel to the electrode wall, becomes unstable.

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